ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 324 (2006) 425–434
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Extremal solutions of second order impulsive dynamic equations on time scales

M. Benchohra[a], S.K. Ntouyas[b,*], A. Ouahab[a]

[a] *Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algérie*
[b] *Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*

Received 28 March 2005
Available online 9 January 2006
Submitted by A.C. Peterson

Abstract

In this paper, we give an existence theorem for the extremal solutions for second order impulsive dynamic equations on time scales.

Keywords: Impulsive dynamic equations; Delta derivative; Fixed point; Time scale; Extremal solution

1. Introduction

This paper is concerned with the existence of extremal solutions of second order impulsive dynamic equations on time scales. More precisely, we consider the following initial value problem:

$$-y^{\Delta\Delta}(t) = f\big(t, y(t)\big), \quad t \in J := [0,b] \cap \mathbb{T},\ t \neq t_k,\ k = 1, \dots, m, \tag{1}$$

$$y\big(t_k^+\big) - y\big(t_k^-\big) = I_k\big(y\big(t_k^-\big)\big), \quad k = 1, \dots, m, \tag{2}$$

$$y^{\Delta}\big(t_k^+\big) - y^{\Delta}\big(t_k^-\big) = \bar{I}_k\big(y\big(t_k^-\big)\big), \quad k = 1, \dots, m, \tag{3}$$

$$y(0) = y_0, \quad y^{\Delta}(0) = y_1, \tag{4}$$

* Corresponding author.
E-mail addresses: benchohra@univ-sba.dz (M. Benchohra), sntouyas@cc.uoi.gr (S.K. Ntouyas), ouahab@univ-sba.dz (A. Ouahab).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.12.028

where $\mathbb{T}$ is a time scale, $f:[0,b]\times\mathbb{R}\to\mathbb{R}$ is a given function, $I_k, \bar{I}_k:\mathbb{R}\to\mathbb{R}$, $t_k\in\mathbb{T}$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=b$, $y(t_k^+)=\lim_{h\to 0^+} y(t_k+h)$ and $y(t_k^-)=\lim_{h\to 0^+} y(t_k-h)$ represent the right and left limits of $y(t)$ at $t=t_k$.

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs of Lakshmikantham et al. [10], and Samoilenko and Perestyuk [12] and the references therein.

The theory of dynamic equations on time scales has been developing rapidly and have received much attention in recent years. The study unifies existing results in differential and finite difference equations, and provides powerful new tools for exploring connections between the traditionally separated fields. We refer to the books by Bohner and Peterson [6,7], Lakshmikantham et al. [11] and to the papers cited therein.

The time scales calculus has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics, neural networks, social sciences, see the monographs of Aulbach and Hilger [2], Bohner and Peterson [6,7], Lakshmikantham et al. [11] and to the references therein.

Very recently, impulsive dynamic equations on time scales have been studied by Agarwal et al. [1], Benchohra et al. [4,5], and Henderson [8]. In [3] the existence of extremal solutions for a class of first order impulsive dynamic equations is considered under the usual order on lower and upper solutions. In this paper we shall prove our existence theorem for the problem (1)–(4) by using the Heikkilä and Lakshmikantham fixed point theorem [9] about the existence of the least and the greatest fixed points for an operator defined on an order interval and with a reversed order between lower and upper solutions. To our best knowledge, the question of the existence of extremal solutions for second order impulsive dynamic equations on time scales has not been yet considered. Hence, these results can be considered as a contribution to this field.

2. Preliminaries

We will briefly recall some basic definitions and facts from the time scales calculus that we will use in the sequel.

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}:\ s>t\}\quad\text{and}\quad\rho(t)=\sup\{s\in\mathbb{T}:\ s<t\}$$

(supplemented by $\inf\emptyset:=\sup\mathbb{T}$ and $\sup\emptyset:=\inf\mathbb{T}$) are well defined. The point $t\in\mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t)=t$, $\rho(t)<t$, $\sigma(t)=t$, $\sigma(t)>t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k:=\mathbb{T}-\{m\}$; otherwise, set $\mathbb{T}_k=\mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k:=\mathbb{T}-\{M\}$; otherwise, set $\mathbb{T}^k=\mathbb{T}$. The notations $[c,d]$, $[c,d)$, and so on, will denote time scales intervals such as

$$[c,d]=\{t\in\mathbb{T}:\ c\leqslant t\leqslant d\},$$

where $c,d\in\mathbb{T}$ with $c<\rho(d)$.

Definition 2.1. Let X be a Banach space. The function $f:\mathbb{T}\to X$ is called *rd-continuous* provided it is continuous at each right-dense point and has a left-sided limit at each point; write $f\in C_{\mathrm{rd}}(\mathbb{T})=C_{\mathrm{rd}}(\mathbb{T},X)$.

For $t\in\mathbb{T}^k$, let the Δ derivative of f at t, denoted $f^{\Delta}(t)$, be the number (provided it exists), such that for all $\varepsilon>0$ there exists a neighborhood U of t such that

$$\left|f\big(\sigma(t)\big)-f(s)-f^{\Delta}(t)\big[\sigma(t)-s\big]\right|\leqslant\varepsilon\left|\sigma(t)-s\right|$$

for all $s\in U$.

A function F is called an antiderivative of $f:\mathbb{T}\to X$ provided

$$F^{\Delta}(t)=f(t)\quad\text{for each } t\in\mathbb{T}^k.$$

$C([0,b],\mathbb{R})$ is the Banach space of all continuous functions from $[0,b]$ into $\mathbb{R}$ where $[0,b]\subset\mathbb{T}$ with the norm

$$\|y\|_{\infty}=\sup\big\{\left|y(t)\right|:\ t\in[0,b]\big\}.$$

Remark 2.1. Let $L^1([0,b],\mathbb{R})$ represents the space of functions which are Lebesgue integrable in the time scale sense and $AC^i([0,b],\mathbb{R})$ be the space of i-times differentiable functions $y:[0,b]\to\mathbb{R}$ whose ith delta derivative, $y^{\Delta^{(i)}}$, is absolutely continuous.

(i) If f is continuous, then f is rd-continuous.
(ii) If f is delta differentiable at t, then f is continuous at t.

We use the following fixed point theorem of Heikkilä and Lakshmikantham [9] in the sequel.

Theorem 2.1. *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q:[a,b]\to[a,b]$ be a nondecreasing mapping. If each sequence $\{Qx_n\}\subset Q([a,b])$ converges, whenever $\{x_n\}$ is a monotone sequence in $[a,b]$, then the sequence of Q-iteration of a converges to the least fixed point x_* of Q and the sequence of Q-iteration of b converges to the greatest fixed point x^* of Q. Moreover,*

$$x_*=\min\big\{y\in[a,b]:\ y\geqslant Qy\big\}\quad\text{and}\quad x^*=\max\big\{y\in[a,b]:\ y\leqslant Qy\big\}.$$

Definition 2.2. A mapping $f:J\times\mathbb{R}\to\mathbb{R}$ is said to be L^1-Chandrabhan if

(i) $t\mapsto f(t,x)$ is measurable for each $x\in\mathbb{R}$;
(ii) $x\mapsto f(t,x)$ is nondecreasing in x for almost every $t\in J$;
(iii) for each $q>0$, there exists $h_q\in L^1(J,\mathbb{R}_+)$ such that

$$\left|f(t,x)\right|\leqslant h_q(t)$$

for all $|x|\leqslant q$ and for almost each $t\in J$.

3. Main result

We will assume for the remainder of the paper that, for each $k=1,\dots,m$, the points of impulse t_k are right dense. In order to define the solution of (1)–(4), we shall consider the following space:

$$PC = \{y : [0, b] \to \mathbb{R} \text{ is continuous except at the points } t_k,\ k = 0, \dots, m, \text{ for which } y(t_k^-) \text{ and } y(t_k^+),\ k = 1, \dots, m, \text{ exist with } y(t_k^-) = y(t_k)\},$$

which is a Banach space with the norm

$$\|y\|_{PC} = \sup\{|y(t)|:\ t \in [0, b]\}.$$

Let us start by defining what we mean by a solution of problem (1)–(4).

Definition 3.1. A function $y \in PC \cap AC^1(J \setminus \{t_1, \dots t_m\}, \mathbb{R})$ is said to be a solution of (1)–(4) if it satisfies the differential equation

$$-y^{\Delta\Delta}(t) = f(t, y(t)) \quad \text{a.e. on } J \setminus \{t_k\},\ k = 1, \dots, m,$$

and for each $k = 1, \dots, m$ the function y satisfies the conditions $y(t_k^+) - y(t_k^-) = I_k(y(t_k^-))$, $y^\Delta(t_k^+) - y^\Delta(t_k^-) = \bar{I}_k(y(t_k^-))$ and the initial conditions $y(0) = y_0$ and $y^\Delta(0) = y_1$.

Introduce the following concept of lower and upper solutions for (1)–(4). It will be the basic tool in the approach that follows.

Definition 3.2. A function $\alpha \in PC$ is said to be a lower solution of (1)–(4) if

$$\begin{aligned}
&-\alpha^{\Delta\Delta}(t) \leqslant f(t, \alpha(t)) \quad \text{a.e. on } J,\ t \neq t_k,\\
&\alpha(t_k^+) - \alpha(t_k^-) = I_k(\alpha(t_k^-)), \quad k = 1, \dots, m,\\
&\alpha^\Delta(t_k^+) - \alpha^\Delta(t_k^-) \geqslant \bar{I}_k(\alpha(t_k^-)), \quad k = 1, \dots, m,\\
&\alpha(0) \geqslant y_0 \quad \text{and} \quad \alpha^\Delta(0) \geqslant y_1.
\end{aligned}$$

Similarly, a function $\beta \in PC$ is said to be an upper solution of (1)–(3) if

$$\begin{aligned}
&-\beta^{\Delta\Delta}(t) \geqslant f(t, \beta(t)) \quad \text{a.e. on } J,\ t \neq t_k,\\
&\beta(t_k^+) - \beta(t_k^-) = I_k(\beta(t_k^-)), \quad k = 1, \dots, m,\\
&\beta^\Delta(t_k^+) - \beta^\Delta(t_k^-) \leqslant \bar{I}_k(\beta(t_k^-)), \quad k = 1, \dots, m,\\
&\beta(0) \leqslant y_0 \quad \text{and} \quad \beta^\Delta(0) \leqslant y_1.
\end{aligned}$$

We need the following auxiliary result.

Lemma 3.1. *Let $y_0, y_1 \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$ be rd-continuous and regressive. Then y is the unique solution of the initial value problem*

$$-y^{\Delta\Delta}(t) = f(t), \tag{5}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{6}$$

$$y^\Delta(t_k^+) - y^\Delta(t_k^-) = \bar{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{7}$$

$$y(0) = y_0, \quad y^\Delta(0) = y_1, \tag{8}$$

if and only if

$$y(t) = y_0 + t y_1 - \int_0^t (t-s) f(s)\,\Delta s + \int_0^t \mu(s) f(s)\,\Delta s + \sum_{0<t_k<t} \left[I_k\left(y\left(t_k^-\right)\right) + (t-t_k)\bar{I}_k\left(y\left(t_k^-\right)\right)\right]. \tag{9}$$

Proof. Let y be a solution of the problem (5)–(8). Then

$$-y^{\Delta\Delta}(t) = f(t) \quad \text{for } t \in [0, t_1] \subset \mathbb{T}.$$

An integration from 0 to t (here $t \in (0, t_1]$) of both sides of the above equality yields

$$-\int_0^t y^{\Delta\Delta}(s)\,\Delta s = \int_0^t f(s)\,\Delta s,$$

$$-y^{\Delta}(t) + y^{\Delta}(0) = \int_0^t f(s)\,\Delta s.$$

Thus for $t \in [0, t_1]$, we have

$$-y^{\Delta}(t) = -y^{\Delta}(0) + \int_0^t f(s)\,\Delta s.$$

We integrate both sides of the above equality to get

$$-y(t) + y(0) = -t y_1 + \int_0^t \int_0^s f(u)\,\Delta u\,\Delta s = -t y_1 + \int_0^t (t-s) f(s)\,\Delta s - \int_0^t \mu(s) f(s)\,\Delta s.$$

Then for $t \in [0, t_1]$, we have

$$y(t) = y_0 + t y_1 + \int_0^t (s-t) f(s)\,\Delta s + \int_0^t \mu(s) f(s)\,\Delta s.$$

If $t \in (t_1, t_2]$, then we have

$$-\int_0^t y^{\Delta\Delta}(s)\,\Delta s = \int_0^t f(s)\,\Delta s,$$

$$\int_0^{t_1} y^{\Delta\Delta}(s)\,\Delta s - \int_{t_1}^t y^{\Delta\Delta}(s)\,\Delta s = \int_0^t f(s)\,\Delta s,$$

$$-y^{\Delta}(t_1) + y^{\Delta}(0) - y^{\Delta}(t) + y^{\Delta}\left(t_1^+\right) = \int_0^t f(s)\,\Delta s,$$

$$-y^{\Delta}(t) + \bar{I}_1\left(y\left(t_1^-\right)\right) + y_1 = \int_0^t f(s)\,\Delta s.$$

An integration from t_1 to t of the both sides of the above equality yields

$$\begin{aligned}
&\int_{t_1}^{t}\left[-y^{\Delta}(s)+\bar{I}_1\left(y\left(t_1^-\right)\right)+y_1\right]\Delta s=\int_{t_1}^{t}\int_{0}^{s} f(u)\,\Delta u\,\Delta s,\\
&-y(t)+y\left(t_1^+\right)+(t-t_1)\bar{I}_1\left(y\left(t_1^-\right)\right)+(t-t_1)y_1=\int_{t_1}^{t}\int_{0}^{s} f(u)\,\Delta u\,\Delta s,\\
&-y(t)+y\left(t_1^+\right)+(t-t_1)\bar{I}_1\left(y\left(t_1^-\right)\right)+(t-t_1)y_1\\
&\quad=\int_0^t tf(s)\,\Delta s-\int_0^{t_1} t_1 f(s)\,\Delta s-\int_{t_1}^{t}\sigma(s)f(s)\,\Delta s.
\end{aligned}$$

Thus for $t\in(t_1,t_2]$, we have

$$\begin{aligned}
-y(t)&=-y\left(t_1^+\right)-(t-t_1)\bar{I}_1\left(y\left(t_1^-\right)\right)-(t-t_1)y_1\\
&\quad+\int_0^t tf(s)\,\Delta s-\int_0^{t_1} t_1 f(s)\,\Delta s\\
&\quad-\int_{t_1}^{t}\mu(s)f(s)\,\Delta s-\int_{t_1}^{t} sf(s)\,\Delta s\\
&=-y\left(t_1^-\right)-I_1\left(y\left(t_1^-\right)\right)-(t-t_1)\bar{I}_1\left(y\left(t_1^-\right)\right)-(t-t_1)y_1\\
&\quad+\int_0^t tf(s)\,\Delta s-\int_0^{t_1} t_1 f(s)\,\Delta s-\int_{t_1}^{t} sf(s)\,\Delta s-\int_{t_1}^{t}\mu(s)f(s)\,\Delta s\\
&=-y_0-t_1y_1+\int_0^{t_1}(t_1-s)f(s)\,\Delta s-\int_0^{t_1}\mu(s)f(s)\,\Delta s\\
&\quad+\int_0^t tf(s)\,\Delta s-\int_0^{t_1} t_1 f(s)\,\Delta s-\int_{t_1}^{t} sf(s)\,\Delta s-\int_{t_1}^{t}\mu(s)f(s)\,\Delta s\\
&\quad-I_1\left(y\left(t_1^-\right)\right)-(t-t_1)\bar{I}_1\left(y\left(t_1^-\right)\right)-(t-t_1)y_1.
\end{aligned}$$

Hence for $t\in[t_1,t_2]$, we have

$$y(t)=y_0+ty_1-\int_0^t(t-s)f(s)\,\Delta s+\int_0^t\mu(s)f(s)\,\Delta s+I_1\left(y\left(t_1^-\right)\right)+(t-t_1)\bar{I}_1\left(y\left(t_1^-\right)\right).$$

Continue to obtain for $t\in[0,b]$ that

$$\begin{aligned}
y(t)&=y_0+ty_1-\int_0^t(t-s)f(s)\,\Delta s+\int_0^t\mu(s)f(s)\,\Delta s\\
&\quad+\sum_{0<t_k<t}\left[I_k\left(y\left(t_k^-\right)\right)+(t-t_k)\bar{I}_k\left(y\left(t_k^-\right)\right)\right].
\end{aligned}$$

Conversely we prove that if y satisfies the integral equation (9), then y is a solution of the problem (5)–(8). Firstly $y(0) = y_0$. Let $t \in [0, b] \setminus \{t_1, \dots, t_m\}$ and

$$y(t) = y_0 + ty_1 - \int_0^t (t-s) f(s)\,\Delta s + \int_0^t \mu(s) f(s)\,\Delta s + \sum_{0<t_k<t} \left[I_k\big(y\big(t_k^-\big)\big) + (t-t_k)\bar{I}_k\big(y\big(t_k^-\big)\big)\right].$$

Then

$$\begin{aligned} y^{\Delta}(t) &= \left[y_0 + ty_1 - \int_0^t (t-s) f(s)\,\Delta s + \int_0^t \mu(s) f(s)\,\Delta s \right.\\ &\qquad \left. + \sum_{0<t_k<t} \left[I_k\big(y\big(t_k^-\big)\big) + (t-t_k)\bar{I}_k\big(y\big(t_k^-\big)\big)\right]\right]^{\Delta} \\ &= [y_0 + ty_1]^{\Delta} - \left[\int_0^t (t-s) f(s)\,\Delta s\right]^{\Delta} + \left[\int_0^t \mu(s) f(s)\,\Delta s\right]^{\Delta} \\ &\qquad + \left[\sum_{0<t_k<t} \left[I_k\big(y\big(t_k^-\big)\big) + (t-t_k)\bar{I}_k\big(y\big(t_k^-\big)\big)\right]\right]^{\Delta} \\ &= y_1 - \int_0^t f(s)\,\Delta s - \sigma(t) f(t) + t f(t) + \mu(t) f(t) + \sum_{0<t_k<t} \bar{I}_k\big(y\big(t_k^-\big)\big) \\ &= y_1 - \int_0^t f(s)\,\Delta s + \sum_{0<t_k<t} \bar{I}_k\big(y\big(t_k^-\big)\big). \end{aligned}$$

Thus

$$y^{\Delta\Delta}(t) = \left[y_1 - \int_0^t f(s)\,\Delta s + \sum_{0<t_k<t} \bar{I}_k\big(y\big(t_k^-\big)\big)\right]^{\Delta} = -f(t).$$

Then

$$-y^{\Delta\Delta}(t) = f(t), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}.$$

Clearly, we have $y^{\Delta}(0) = y_1$ and

$$y^{\Delta}\big(t_k^+\big) - y^{\Delta}\big(t_k^-\big) = \bar{I}_k\big(y\big(t_k^-\big)\big), \quad \text{for } k = 1, \dots, m.$$

From the definition of y we can prove that

$$y\big(t_k^+\big) - y\big(t_k^-\big) = I_k\big(y\big(t_k^-\big)\big), \quad \text{for } k = 1, \dots, m. \qquad \square$$

Theorem 3.1. *Assume that hypotheses*

(A1) *the function f is L^1-Chandrabhan*;

(A2) *the functions* I_k, $\bar{I}_k$, $k = 1, \dots, m$, *are continuous and nondecreasing*;
(A3) *there exist* α *and* $\beta \in PC$, *lower and upper solutions, respectively, for the problem* (1)–(4), *such that* $\beta \leqslant \alpha$,

are satisfied. Then the problem (1)–(4) *has a minimal and a maximal solution.*

Proof. Consider the operator $G : PC \to PC$ defined by

$$G(y)(t) = y_0 + t y_1 - \int_0^t \bigl(t - \sigma(s)\bigr) f\bigl(s, y(s)\bigr)\, \Delta s + \sum_{0<t_k<t} \bigl[I_k\bigl(y\bigl(t_k^-\bigr)\bigr) + (t - t_k)\bar{I}_k\bigl(y\bigl(t_k^-\bigr)\bigr)\bigr].$$

It is clear from Lemma 3.1 the fixed points of G are solutions to (1)–(4).

We shall show that G satisfies the assumptions of Theorem 2.1. We prove that $G([\beta, \alpha]) \subseteq [\beta, \alpha]$. Let $y \in [\beta, \alpha]$ and $t \in [0, b]$, and

$$t_i = \max\{t_k\colon\ t_k < t\}.$$

By the definition of the upper solution and the conditions (A1)–(A3), we get

$$\begin{aligned}
G(y)(t) \geqslant{}& \beta(0) + t\beta^{\Delta}(0) - \int_0^{t_1} \bigl(t - \sigma(s)\bigr) f\bigl(s, \beta(s)\bigr)\, \Delta s \\
&- \int_{t_1^+}^{t_2} \bigl(t - \sigma(s)\bigr) f\bigl(s, \beta(s)\bigr)\, \Delta s - \cdots - \int_{t_i}^{t} \bigl(t - \sigma(s)\bigr) f\bigl(s, \beta(s)\bigr)\, \Delta s \\
&+ \sum_{k=1}^{i} \bigl[I_k\bigl(\beta\bigl(t_k^-\bigr)\bigr) + (t - t_k)\bar{I}_k\bigl(\beta\bigl(t_k^-\bigr)\bigr)\bigr] \\
\geqslant{}& \beta(0) + t\beta^{\Delta}(0) + \int_0^{t_1} \bigl(t - \sigma(s)\bigr)\beta^{\Delta\Delta}(s)\, \Delta s \\
&+ \int_{t_1^+}^{t_2} \bigl(t - \sigma(s)\bigr)\beta^{\Delta\Delta}(s)\, \Delta s + \cdots + \int_{t_i^+}^{t} \bigl(t - \sigma(s)\bigr)\beta^{\Delta\Delta}(s)\, \Delta s \\
&+ \sum_{k=1}^{i} \bigl[I_k\bigl(\beta\bigl(t_k^-\bigr)\bigr) + (t - t_k)\bar{I}_k\bigl(\beta\bigl(t_k^-\bigr)\bigr)\bigr].
\end{aligned}$$

Since

$$\bigl[s\beta^{\Delta}(s)\bigr]^{\Delta} = \beta^{\Delta}(s) + \sigma(s)\beta^{\Delta\Delta}(s) \quad \Longrightarrow \quad \sigma(s)\beta^{\Delta\Delta}(s) = \bigl[s\beta^{\Delta}(s)\bigr]^{\Delta} - \beta^{\Delta}(s),$$

then we have

$$
\begin{aligned}
G(y)(t) &\geqslant \beta(0) + t\beta^{\Delta}(0) + \int_0^{t_1} t\beta^{\Delta\Delta}(s)\,\Delta s \\
&\quad + \int_0^{t_1} \bigl(\beta^{\Delta}(s) - \bigl[s\beta^{\Delta}(s)\bigr]^{\Delta}\bigr)\,\Delta s + \int_{t_1^+}^{t_2} t\beta^{\Delta\Delta}(s)\,\Delta s + \int_{t_1^+}^{t_2} \bigl(\beta^{\Delta}(s) - \bigl[s\beta^{\Delta}(s)\bigr]^{\Delta}\bigr)\,\Delta s \\
&\quad + \cdots + \int_{t_i^+}^{t} t\beta^{\Delta\Delta}(s)\,\Delta s + \int_{t_i^+}^{t} \bigl(\beta^{\Delta}(s) - \bigl[s\beta^{\Delta}(s)\bigr]^{\Delta}\bigr)\,\Delta s \\
&\quad + \sum_{k=1}^{i} \bigl[I_k\bigl(\beta\bigl(t_k^-\bigr)\bigr) + (t-t_k)\bar{I}_k\bigl(\beta\bigl(t_k^-\bigr)\bigr)\bigr] \\
&= \beta(0) + t\beta^{\Delta}(0) + t\beta^{\Delta}\bigl(t_1^-\bigr) - t\beta^{\Delta}(0) + \beta\bigl(t_1^-\bigr) - \beta(0) - t_1\beta^{\Delta}\bigl(t_1^-\bigr) \\
&\quad + t\beta^{\Delta}\bigl(t_2^-\bigr) - t\beta^{\Delta}\bigl(t_1^+\bigr) + \beta\bigl(t_2^-\bigr) - \beta\bigl(t_1^+\bigr) - t_2\beta^{\Delta}\bigl(t_2^-\bigr) + t_1\beta^{\Delta}\bigl(t_1^+\bigr) \\
&\quad + \cdots + t\beta^{\Delta}(t) - t\beta^{\Delta}\bigl(t_i^+\bigr) + \beta(t) - \beta\bigl(t_i^+\bigr) - t\beta^{\Delta}(t) + t_i\beta^{\Delta}\bigl(t_i^+\bigr) \\
&\quad + \sum_{k=1}^{i} \bigl[I_k\bigl(\beta\bigl(t_k^-\bigr)\bigr) + (t-t_k)\bar{I}_k\bigl(\beta\bigl(t_k^-\bigr)\bigr)\bigr] \\
&= (t-t_1)\beta^{\Delta}\bigl(t_1^-\bigr) + \beta\bigl(t_1^-\bigr) - \beta\bigl(t_1^+\bigr) + (t-t_2)\beta^{\Delta}\bigl(t_2^-\bigr) - (t-t_1)\beta^{\Delta}\bigl(t_1^+\bigr) + \beta\bigl(t_2^-\bigr) \\
&\quad + \cdots + \beta(t) + \beta\bigl(t_i^-\bigr) - \beta\bigl(t_i^+\bigr) + \sum_{k=1}^{i} \bigl[I_k\bigl(\beta\bigl(t_k^-\bigr)\bigr) + (t-t_k)\bar{I}_k\bigl(\beta\bigl(t_k^-\bigr)\bigr)\bigr] \\
&= -(t-t_1)\bigl(\beta^{\Delta}\bigl(t_1^+\bigr) - \beta^{\Delta}\bigl(t_1^-\bigr)\bigr) - \bigl(\beta\bigl(t_1^+\bigr) - \beta\bigl(t_1^-\bigr)\bigr) \\
&\quad - (t-t_2)\bigl(\beta^{\Delta}\bigl(t_2^+\bigr) - \beta^{\Delta}\bigl(t_2^-\bigr)\bigr) - \bigl(\beta\bigl(t_2^+\bigr) - \beta\bigl(t_2^-\bigr)\bigr) \\
&\quad - \cdots + \beta(t) - \bigl(\beta\bigl(t_i^+\bigr) - \beta\bigl(t_i^-\bigr)\bigr) + \sum_{k=1}^{i} \bigl[I_k\bigl(\beta\bigl(t_k^-\bigr)\bigr) + (t-t_k)\bar{I}_k\bigl(\beta\bigl(t_k^-\bigr)\bigr)\bigr] \\
&\geqslant -(t-t_1)\bar{I}_1\bigl(\beta\bigl(t_1^-\bigr)\bigr) - I_1\bigl(\beta\bigl(t_1^-\bigr)\bigr) - (t-t_2)\bar{I}_2\bigl(\beta\bigl(t_1^-\bigr)\bigr) - I_2\bigl(\beta\bigl(t_2^-\bigr)\bigr) \\
&\quad + \cdots + \beta(t) - \bar{I}_i\bigl(\beta\bigl(t_i^-\bigr)\bigr) - I_i\bigl(\beta\bigl(t_i^-\bigr)\bigr) + \sum_{k=1}^{i} \bigl[I_k\bigl(\beta\bigl(t_k^-\bigr)\bigr) + (t-t_k)\bar{I}_k\bigl(\beta\bigl(t_k^-\bigr)\bigr)\bigr] \\
&= \beta(t).
\end{aligned}
$$

Thus $G(y)(t) \geqslant \beta(t)$, $t \in [0,b]$. By an analogous relation, obtained by replacing α by β, and changing the above inequality it follows that $G(y)(t) \leqslant \alpha(t)$, $t \in [0,b]$. Hence

$$\beta \leqslant G(y) \leqslant \alpha \quad \Longrightarrow \quad \bigl\|G(y)\bigr\|_{PC} \leqslant \max\bigl(\|\alpha\|_{PC}, \|\beta\|_{PC}\bigr).$$

Finally, let $\{y_n\}_{n\in\mathbb{N}}$ be a monotone sequence in $[\beta,\alpha]$. We shall show that the sequence $G(y_n)(t)$ converges in $G([\beta,\alpha])$. Since $N(y_n) \in [\beta,\alpha]$. Thus

$$\bigl\|G(y_n)\bigr\|_{PC} \leqslant \max\bigl(\sup\bigl\{\bigl|\alpha(t)\bigr|\colon t \in [0,b]\bigr\},\ \sup\bigl\{\bigl|\beta(t)\bigr|\colon t \in [0,b]\bigr\}\bigr) := q < \infty.$$

By (A1)–(A3) we can easily show that if

$$x \leqslant y \quad \Longrightarrow \quad G(x) \leqslant G(y).$$

Let $\{y_n\}_{n\in\mathbb{N}}$ be a monotone sequence in $[\beta,\alpha]$. We shall show that $\{G(y_n)\}_{n\in\mathbb{N}}$ is an equicontinuous set. Let $r_1, r_2 \in [0,b]$, $r_1 < r_2$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}
\bigl|(Gy)(r_2)-(Gy)(r_1)\bigr| &\leqslant |r_2-r_1||y_1| + \int_0^{r_1}(r_2-r_1)\bigl|f\bigl(s,y(s)\bigr)\bigr|\,\Delta s \\
&\quad + \int_{r_1}^{r_2} r_2\bigl|f\bigl(s,y(s)\bigr)\bigr|\,\Delta s + \int_{r_1}^{r_2}\bigl|\mu(s)\bigr|\bigl|f\bigl(s,y(s)\bigr)\bigr|\,\Delta s \\
&\quad + \sum_{0<t_k<r_2-r_1}\Bigl[\sup_{x\in[-q,q]}\bigl|I_k(x)\bigr| + (r_2-r_1)\sup_{x\in[-q,q]}\bigl|\bar{I}_k(x)\bigr|\Bigr] \\
&\leqslant |r_1-r_2||y_1| + \int_0^{r_1}|r_1-r_2|h_q(s)\,ds + \int_{r_1}^{r_2}h_q(s)\,ds \\
&\quad + \sum_{0<t_k<r_2-r_1}\Bigl[\sup_{x\in[-q,q]}\bigl|I_k(x)\bigr| + (r_2-r_1)\sup_{x\in[-q,q]}\bigl|\bar{I}_k(x)\bigr|\Bigr].
\end{aligned}$$

The right-hand side tends to zero as $r_2 - r_1 \to 0$. Hence $\{G(y_n)\}_{n\in\mathbb{N}}$ is an equicontinuous set and consequently $\{G(y_n)\}_{n\in\mathbb{N}}$ is relatively compact by Arzelá–Ascoli theorem. We can conclude that $\{G(y_n)\}_{n\in\mathbb{N}}$ converges in $G([\beta,\alpha])$.

As a consequence of Theorem 2.1, we deduce that G has a least and a greatest fixed point in $[\beta,\alpha]$. This further implies that the problem (1)–(4) has minimal and maximal solutions on $[0,b]$. $\square$

References

[1] R.P. Agarwal, M. Benchohra, D. O'Regan, A. Ouahab, Second order impulsive dynamic equations on time scales, Funct. Differ. Equ. 11 (2004) 223–234.
[2] B. Aulbach, S. Hilger, Linear dynamical processes with inhomogeneous time scale, in: Nonlinear Dynamics and Quantum Dynamical Systems, Akademie Verlage, Berlin, 1990.
[3] A. Belarbi, M. Benchohra, A. Ouahab, Extremal solutions for impulsive dynamic equations on time scales, Comm. Appl. Nonlinear Anal. 12 (2005) 85–95.
[4] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, On first order impulsive dynamic equations on time scales, J. Difference Equ. Appl. 10 (2004) 541–548.
[5] M. Benchohra, S.K. Ntouyas, A. Ouahab, Existence results for second order boundary value problem for impulsive dynamic equations on time scales, J. Math. Anal. Appl. 296 (2004) 69–73.
[6] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, New York, 2001.
[7] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[8] J. Henderson, Double solutions of impulsive dynamic boundary value problems on a time scale, J. Difference Equ. Appl. 8 (2002) 345–356.
[9] S. Heikkilä, V. Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, Dekker, New York, 1994.
[10] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[11] V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, Dynamic Systems on Measure Chains, Kluwer Acad. Publ., Dordrecht, 1996.
[12] A.M. Samoilenko, N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.